One can hardly find an area of physical and technical research, or a branch of mathematics that does not use complex numbers and the theory of analytic functions of a complex variable. The latter has a wide and diverse range of applications in spite of its significant disadvantage, which lies in the fact that the known complex numbers and functions are two-dimensional and their applications are restricted by "plane" problems. The real space is three-dimensional, and moreover - according to the special theory of relativity - physical space with time forms the four-dimensional real pseudo-Euclidean so-called "Minkowski world" with the metric

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where – dimensional orthogonal Cartesian coordinates, and *t*–‍time (– speed of light in vacuum).

The attempts made (mainly in the 19-th century) to generalize real and complex numbers two-dimensional for a space with greater number of dimensions have failed. The introduction of new, hyper complex numbers (in particular, quaternions) is invariably accompanied by the loss of one of the basic arithmetic functions - commutativity of these numbers when multiplied3,4. And so far in mathematics there is a very strong and seemingly well-founded assertion that three-dimensional and four-dimensional complex numbers similar in their arithmetic properties to two-dimensional complex numbers do not exist. This situation, for example, is clearly articulated in a concise and meaningful foreword to one of the university courses on the theory of functions of the complex variable 5: "Trenchant studies conducted by K.T.W. Weierstrass (1815-1897), F.G. Frobenius (1849-1917) and C. Peirce showed that any generalization of complex numbers inevitably violates one of the laws of arithmetic operations. Thus, it appears that the only generalization of real numbers which preserves the known laws of arithmetic is complex numbers"(meaning - two-dimensional).

However, understanding how deeply values and functions permeate the basic schemes of modern physics, in particular - quantum mechanics, the theory of wave and quantum fields ... (for this it is enough to refer to the Hermitean operators describing the components of momentum and angular momentum of elementary particles, the waves de‑Broglie, Schrödinger equation ... etc) - it is not easy to immediately and unconditionally agree to the lack of complex numbers and functions, proportionate to the real four-dimensional space-time, i.e., to accept such a "mismatch" between the mathematical apparatus and properties of physical objects described by them. It is quite possible that the wave-particle nature of elementary particles (their behavior in the microcosm) is in the best description of their conditions by four-dimensional analytic wave functions of the complex variable.

Indeed, as shown in the second chapter, four-dimensional complex numbers exist. Moreover, in addition to the above, the introduction of four-dimensional complex numbers, the set of which *adequately* reflects the real space-time (i.e., forms a complex space, similar in geometrical properties to the Minkowski world), is of great interest in terms of application of analytic functions of a complex four-dimensional variable to the macroscopic problems of electrodynamics, hydro- (and aero-) dynamics. It is possible that at speeds commensurable with the speed of light, the description, for example, of aerodynamic phenomena with the use of four-dimensional complex functions will be more preferable than their description on the basis of two-dimensional complex functions.

Several new specific numbers of multidimensional concepts and methods were used in the analysis of multidimensional (three- and four-dimensional) complex functions that were not used for functions of two-dimensional variable were tentatively broadly defined in the first chapter, generally having an introductory character. However, specialists in mathematical physics can skim through the first chapter, so to say, "flip through" it, paying attention only to the considerations set out at the end of the chapter. Then one can proceed to read the primary, second chapter.

On the other hand, it should be noted that the introduction has a certain heuristic value, because it reflects the direct, original (one might say, chronological) way of thinking that led to the discovery of three- and four-dimensional complex numbers and analytic functions of corresponding complex variables.

Four-dimensional complex (or abbreviated, 4D-) numbers  are defined and analyzed in the first three sections of the second chapter. In particular, the defined characters *i1 i2* and *i3* appearing in them instead of the Euler imaginary units  - actual components, ). Namely, we set quadratic relations (6.25) linking these characters together. We obtained important representations (6.23) of characters in the form of square matrices of the fourth grade. We formulated the rules of correlations (6.25) of multiplication (6.29) of 4D-numbers ... (and other results).

The conditions under which the complex *х*-space formed by 4D-numbers, adequately reflects the pseudo-Euclidean Minkowski world. At the same time, we revealed the fundamental connection of the structure of 4D-numbers with the pseudo-Euclidean geometry of the real space-time underlying the special theory of relativity.

Later, we laid the foundation of the theory of analytic functions of the 4D-variable was. Namely, we derived the necessary and sufficient conditions for the differentiability (or analyticity) of these functions, similar to the conditions of D'Alembert-Euler for the functions of 2D-variable. It was shown that analytical 4D-function has derivatives of all orders. And the sum (difference) and multiplication of 4D-analytical functions are also analytic functions.

We examined the infinite series of 4D-numbers (and the properties of such absolutely convergent series), as well as infinite power series of 4D-variable (and the properties of such series at their absolute and uniform convergence). We defined on the basis of the expansions in power series the basic 4D-analytical functions (exponential, trigonometric and hyperbolic) and confirmed their basic properties.

We have proven two integral theorems representing an analog of the fundamental Cauchy theorem. The first one states that the integral (10.12) in a closed circuit C from the unique 4D-function *f*(*q*), which is analytical on the surface (and in the immediate vicinity of it) enveloped by this circuit equals to zero. The second theorem is that the integral (10.21) in a closed hypersurface σ from the valued 4D-function *f*(*q*), analytic in the four-dimensional volume covered by this hypersurface, also equals to zero.

In addition, we derived a formula (similar to the well-known Cauchy formula), which is expressed in the form (10.36), the value of 4K-function *f*(*q*), analytic on the surface S', at any *а* point of this surface through the values of the same function on the contour C \*, limiting S'. Additionally we presented the corresponding formula (10.37) for the derivatives of the random order from the analytic *f* function on the 4D-variable. Further, we formulated the expansions of analytic 4D-functions in power series similar to Taylor and Laurent series for 2D-functions. We found out the conditions of termwise differentiation and integration of infinite series of analytic 4D-functions. The book presents a number of considerations that should be taken into account when considering the conformal transformations carried out by 4D-analytical functions mainly in *х*-representation and during the calculation of deductions of 4D-functions in singular points.

In addition, it presents (later on called Euclidean ) 4D-numbers on the set as determined by analytical 4D-functions subordinate to four-dimensional Laplace equation (6.42), i.e. the set which adequately reflects the 4-dimensional real Euclidean space with the metric (6.43). And the description of the Euclidean 4D-numbers and 4D-functions directly follows from the description of pseudo-Euclidean 4D-numbers and 4D-functions related to the real 4-dimensional space-time with the metric (5.1) by simply replacing their second component  and  on the components  and  of the Euclidean 4D-numbers and 4D-functions in formulas (6.27) - (6.31) (6.34) (6.36) (6.38) - (6.40 a), (7.1) - (7.5) (7.7) - (7.8 a) etc.. It should also be noted that the Euclidean 4D-numbers having all the commutative, associative and distributive properties typical for arithmetic operations with real numbers (and meeting a number of mandatory requirements), must replace the Hamilton quaternions, the multiplication of which is not subordinate to the law of commutation.

The third chapter outlines possible approaches to the description of the classical wave fields using 4D analytic functions. Namely, two variants of four-dimensional complex potential for an electromagnetic field in void, entered on the basis of the two options of 4 potential for this field in a real pseudo-Euclidean Minkowski world. Accordingly, the components *En* and *Hn* of electric and magnetic strength (*n*=1, 2, 3) are expressed through the first derivatives 4D potential on the coordinate  in two kinds (14.12) and (14.31). Further, by means of analytical 4D-functions we described velocity and pressure fields, resulting in the propagation of acoustic waves in an ideal (liquid and gas) environment.

In the fourth chapter, on the basis of analytical 4D-functions we developed the relativistic quantum mechanical theory of hydrogen-like atom, similar to the well-known Dirac theory. We formulated Lorentz covariant spinor wave equation for a system of two oppositely charged elementary particles, considered in an inertial frame of reference with the origin at the center of mass of the particles, or, more precisely, the equation for a particle with the reduced mass μ, a charge (−e) and the spin , located in the central (Coulomb) electrostatic field of a point charge Ze (Z - atomic number of the nucleus). We formulated Lagrange formalism for this spinor equation and in view of the isotropic conditions of the problem to be solved, set the law of conservation of total angular momentum according to E. Noether's theorem; In particular, the operators of the orbital, spinor and total angular momentum of the electron. It was also shown that if the operators of physical quantities, which are not time-dependent, commute with the Hamiltonian (18.5), corresponding to the spinor equation (17.1), then the mean values of these quantities are integrals of motion. On the basis of this provision, the stationary states of a hydrogen-like atom can be expressed through common eigenfunctions of the square operators of the total angular momentum, its projection on any coordinate axis and the operator of the parity which allows to divide in the spinor wave equation (20.1) the temporal and spatial spherical variables and then re-divide the radial and angular variables. Solutions of radial equations, limited in the range of the radial variable

 

(in particular, if ) and turning to zero when , exist only for energy eigenvalues subordinate to the Dirac equation (20.40). It should be noted that the compliance of the theory presented in the fourth chapter with Dirac theory indirectly confirms the right choice of 4D-numbers made in this book.

Attachment A considers the first kind of two-dimensional (it being - the known complex numbers) and two-dimensional numbers of the second kind, the module of which corresponds to pseudo-Euclidean metric of the space-time plane. It is helpful to keep the latter in mind when discussing a number of general properties of multi-dimensional complex numbers.

Attachment C gives a rigorous but short description of 3-dimensional complex numbers and analytic functions of 3D-variables as quadratic relations (B1.66) for the characters  and  appearing in them do not have the canonical form. We presented the proof of two integral theorems that generalize Cauchy's theorem for 2D functions to 3D functions, as well as proof of generalization of (B3.28) Cauchy's integral formula as the main outcomes in addition to the rules (B1.71) of multiplying of 3D-numbers, analyticity conditions (B2.2) of 3D functions and conditions (B1.26), (B1.28) and (B1.29) adequate display (when æ= 1) in complex *x*-space of the pseudo-Euclidean Minkowski metric subspace (B1.20).

In conclusion, it should be noted that paragraphs marked with an asterisk contain only original drafts on the subjects marked in their names (but, nevertheless, include a number of useful and important relations).